Risk Measures Respecting Comparative Risk Aversion

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January 15, 2009

Abstract

Decisions involving uncertainty depend on two distinct aspects: (i) the risk of the position and (ii) the attitude towards risk of the investor. The literature captures the first aspect by risk measures and the second by risk aversion. We connect both concepts by introducing the class of risk measures which respect comparative risk aversion. The connection is achieved by an axiom, which dates back to Aumann and Serrano (2008) and asserts that less risk averse agents accept riskier gambles. We characterize this class by a simple equivalent condition. This equivalence provides a representation theorem and a construction method for risk measures of this class.

Key Words: measures of risk, risk aversion, duality, coherent risk measures, expected utility, decision-making under risk.

1 Introduction

Whenever facing a position with uncertain outcomes people are exposed to risk. These positions appear in diverse situations and include e.g. any stock, portfolio, credit, gamble or lottery. Conceptually, whether or not an individual enters such a position depends, following Diamond and Stiglitz (1974), on two distinct aspects:

(i) the objective attributes of the position, in particular how risky it is, and

(ii) the subjective attitude towards risk of the investor, in particular how risk averse she is.

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For both aspects the literature provides sophisticated theories. Concerning the first, it suggests numerous *risk measures*, which quantify the risk of a financial position. Famous examples are Value at Risk, Expected Shortfall, spectral risk measures and risk measures based on moments and expected utility. Since some of these examples expose unfavorable properties (e.g. the commonly used Value at Risk penalizes diversification), axioms of coherence were introduced by Artzner et al. (1999) and Föllmer and Schied (2002) to ensure consistent risk measures. Concerning the second aspect, the classic contribution of Arrow (1965) and Pratt (1964) measures the attitude towards risk by defining a coefficient of *risk aversion* based on the subjective utility function.

This paper aims to connect both concepts. For this purpose we introduce and analyze objective risk measures, which respect comparative risk aversion.

The connection is established by postulating an axiom for risk measures, which ensures the consistency with comparative risk aversion. It asserts that every agent accepts any position, which is less risky (by means of the risk measure in question) than a position, that is accepted by a more risk-averse agent. The axiom is powerful, since it applies to any two agents of which one is uniformly more risk averse than the other. Hence it yields risk measures, which are able to dictate the acceptance behavior of almost arbitrary agents in the expected utility framework. More precisely, knowing that some agent accepts a gamble yields that all less risky gambles (easily identified by the total measure) are accepted by all less risk averse agents. It is remarkable that this applies to arbitrary utility functions, hence the risk measure is consistent with two completely different utility functions. This axiom was first formulated in the highly inspiring paper by Aumann and Serrano (2008). By additionally imposing positive homogeneity they uniquely determine a risk measure and analyze it. Since this additional axiom is of minor importance (also in their opinion), we relax it and consider the whole class of risk measures, which satisfy the first axiom.

The crucial key to this class of risk measures is the introduction of the *indifference function*, which maps a position to the parameter of an agent with CARA-utility, who is indifferent between accepting the position or not. The indifference function induces an ordering on the set of positions. It turns out that any risk measure that reverses this ordering respects comparative risk aversion. Hence we derive a method to construct such risk measures by composition of the indifference function and an arbitrary inverting function. Furthermore, we show that in the other direction any respecting risk measures inverses the ordering. Thus we provide a criterion to check for respect to risk aversion. A direct consequence is a representation theorem, which identifies the whole class of respecting risk measures. Overall we provide a handy equivalent condition for respect to risk aversion.

Since the indifference function is defined implicitly and computationally demanding, we propose more explicit expressions by using the means of integral transforms. Moreover, we give first examples of this class. Many of these risk measures satisfy surprisingly many favorable properties and axioms of coherence.

The paper is organized as follows: the following chapter provides the axiom. Section 3 introduce the indifference function, which is crucial for the main results, which are derived in Section 4. Section 5 provides briefly first examples of risk measures respecting risk aversion.

2 Notation and the Axiom

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space. Let $\mathbf{E}$ denote the expectation with respect to $\mathcal{P}$. A *financial position* or *gamble* is described by a random variable $X : \Omega \to \mathbb{R}$. $X(\omega)$ is the discounted net worth of the position under scenario ω. We assume $\mathbf{E}(X) > 0$ and $\mathcal{P}(X < 0) > 0$. Let $\mathfrak{X}$ be the set of financial positions.

Definition 1 (Risk Measure) *A mapping $R : \mathfrak{X} \to \mathbb{R}$ is called a* risk measure*.*

For a given risk measure R, we say a gamble X is riskier than a gamble Y, if $R(X) > R(Y)$.
In the remainder of the section we follow Aumann and Serrano (2008). Let $u : \mathbb{R} \to \mathbb{R}$ denote a Bernoulli utility function for money, which is monotonic, concave and in $\mathcal{C}^2$. Let agent i have utility function u_i and let $w_i \in \mathbb{R}$ be i's wealth level. We say *i accepts X at w*, if $\mathbf{E}u_i(X + w) > u_i(w)$. Else, we say *$i$ rejects X at w*.

Definition 2 (Uniform Comparative Risk Aversion) *Agent* i is as least risk-averse as j *(denoted by $i \trianglerighteq j$), if for all X, w_i, w_j holds the following: agent j accepts X at w_j, if agent i accepts X at w_i.*
Agent i is more risk-averse as j *(denoted by $i \vartriangleright j$), if $i \trianglerighteq j$ and not $j \trianglerighteq i$,*
(or equivalently, if for all X, w_i, w_j holds: agent j accepts X at w_j, if agent i accepts X at w_i and there exist X, w_i, w_j such that i rejects X at w_i and j accepts X at w_j).

Definition 3 (Duality) *A risk measure R is* dual*, if for all agents i, j, gambles X, Y and wealth* *levels w_i, w_j it holds*

$$\text{(1)} \qquad \textit{If } i \rhd j,\ i \textit{ accepts } X \textit{ at } w_i, \textit{ and } R(X) > R(Y), \textit{ then } j \textit{ accepts } Y \textit{ at } w_j.$$

In words, a risk measure is dual, if every agent accepts any gamble, which is less risky (by means of R) than a gamble, which is accepted by a more risk-averse agent. In other words, duality of a risk measure R states that, if the more risk-averse of two agents accepts the riskier (by means of R) of two gambles, than more than ever the less risk-averse agent accepts the less risky gamble. Consequently a dual risk measure respects comparative risk aversion. The two notions of uniform comparative risk aversion and duality were introduced by Aumann and Serrano (2008).

3 CARA and the indifference function

For a utility function u the Arrow-Pratt coefficient of absolute risk aversion is defined by

$$\rho_u(x) = \frac{-u''(x)}{u'(x)}.$$

Any utility function with constant Arrow-Pratt coefficient of absolute risk aversion (CARA) α is up to additive and multiplicative constants of the form $u_\alpha(x) = \frac{1}{\alpha}(1 - e^{-\alpha x})$. The corresponding agent is called CARA with parameter α. Since we only consider risk-averse agents, we concentrate on $\alpha > 0$. The crucial idea in this sections is to consider for a given gamble the parameter α of an CARA-agent, who is indifferent between accepting the gamble or not. Therefore we continue more formally. For any position $X \in \mathfrak{X}$ we define

$$\mathfrak{L}(X, \alpha) := \mathbf{E}e^{-\alpha X} = \int_{\mathbb{R}} e^{-\alpha x} f_X(x)\, dx,$$

where the last equation holds, if X has a continuous density function f_X.

Lemma 4 (Indifference Function) *Defining*

$$\alpha(X) := \begin{cases} \alpha^* & \textit{the positive root of } \mathfrak{L}(X, \alpha^*) = 1, \textit{ if it exists,} \\ \infty & \textit{if } \mathfrak{L}(X, \alpha) < 1 \textit{ holds for all } \alpha > 0, \\ 0 & \textit{else,} \end{cases}$$

yields a well-defined mapping $\alpha : \mathfrak{X} \to \mathbb{R}^+_{0,\infty}$ and is called indifference function.

Proof. Existence is ensured by the definition and uniqueness follows by the convexity of $\mathfrak{L}(X,\alpha)$ in α. To be more precise, we assume α to be not unique for some X, i.e. there exist two positive roots α_1 and α_2 of $\mathfrak{L}(X,\alpha)=1$. W.l.o.g. it is $\alpha_1<\alpha_2$. We consider the second derivative

$$\mathfrak{L}_{\alpha\alpha}(X,\alpha)=\int_{\mathbb{R}} x^2 e^{-\alpha x} f_X(x)\,dx,$$

which is strictly positive or infinite on $\mathbb{R}$ for any position X[1]. This yields that the first derivative $\mathfrak{L}_\alpha(X,\alpha)$ is strictly increasing on $\mathbb{R}$. We show that

$$\mathfrak{L}_\alpha(X,\alpha_1)>0.$$

Else, $\mathfrak{L}_\alpha$ would be negative on $[0,\alpha_1)$ by its strict monotonicity. By $\mathfrak{L}(X,0)=1$ for all $X\in\mathfrak{X}$ this yields $\mathfrak{L}(X,\alpha_1)<1$, which contradicts $\mathfrak{L}(X,\alpha_1)=1$. Finally it is $\mathfrak{L}_\alpha(X,\alpha)>0$ on $[\alpha_1,\alpha_2]$ by its monotonicity and $\mathfrak{L}_\alpha(X,\alpha_1)>0$. This yields $\mathfrak{L}(X,\alpha_2)>1$ and, hence a contradiction. ■

We present an example, in which the indifference function equals zero. This is the case for distributions with too heavy negative tails. The example also shows that the assumptions on position X (positive mean and potential losses) are not sufficient for the existence of a positive root of $\mathfrak{L}(X,\alpha^*)=1$.

Example 5 *Let X have standard normal distribution. We consider the random variable $Y:=2-e^X$. Hence Y is up to additive and multiplicative constants a standard lognormal distribution. Similar to lognormal distribution, which have no moment generating function, in our example the indifference function equals zero. To see this, we consider for any $\alpha>0$*

$$\begin{aligned}\mathbf{E}\exp(-\alpha Y)=\mathbf{E}\exp(\alpha\exp(X)-2\alpha)&=(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty}\exp(\alpha\exp(x)-\frac{x^2}{2}-2\alpha)\,dx\\ &\geq(2\pi)^{-\frac{1}{2}}\int_0^{\infty}\exp(\alpha(1+x+\frac{x^2}{2}+\frac{x^3}{6})-\frac{x^2}{2}-2\alpha)\,dx=\infty.\end{aligned}$$

The inequality follows by omitting integration on the negative axis and by reducing the exponential of x to a third-degree polynomial. Since this polynomial has the positive coefficient α, the integrand tends to infinity for $x\to\infty$. Therefore the indifference function equals zero for any positive α. The assumptions on a position are satisfied, since it holds $\mathbf{E}Y=2-e^{\frac{1}{2}}>0$ and

$$\mathcal{P}(Y<0)=\mathcal{P}(X>\log(2))>0.$$

[1] This holds for any position up to positions which equal zero almost surely. But for these positions α equals zero and is hence unique.

In the remainder of the section, we discuss how to find the parameter, which makes an CARA-agent indifferent between accepting X or not, i.e. the positive root α^* of

$$\mathfrak{L}(X, \alpha^*) = 1.$$

Since solving for the implicitly defined solution is a demanding task, we propose a method, which provides more explicit solutions by the means of integral transforms. Therefore we note that our function $\mathfrak{L}(X, \alpha)$ is just the two-sided Laplace-transform of f_X. By $\mathfrak{L}^{-1}(X, 1)$ we denote the inverse function of $\mathfrak{L}(X, \alpha)$ in argument α mapping into the positive numbers, if it exists. Note that we do not refer to the common inverse of Laplace-transform in argument X yielding the density f_X. We are now able to express α^* in the more explicit form

$$\alpha^*(X) = \mathfrak{L}^{-1}(X, 1).$$

The theory on Laplace transform is highly sophisticated and Laplace-transforms are derived for virtually all distribution. Inverting these Laplace-transforms solves our problem, but it still might be a difficult task. Moreover, there are various integral transform, which help to find α^*. The two sided Laplace transform is closely connected to the one-sided Laplace-transform, denoted by $\mathfrak{L}^1(f_X, \alpha)$, since it holds

$$\mathfrak{L}(X, \alpha) = \mathfrak{L}^1(f_X(x), \alpha) + \mathfrak{L}^1(f_X(-x), -\alpha).$$

The connection to the Fourier transform, denoted by $\mathfrak{F}(f_X, \alpha)$, is given by

$$\mathfrak{L}(X, \alpha) = \mathfrak{F}(f_X, -i\alpha).$$

While inverting the Fourier transform we have to restrict to non-complex solutions. The moment-generating function, which is popular in stochastics and denoted here by $\mathfrak{M}(X, \alpha)$, is also closely connected by

$$\mathfrak{L}(X, \alpha) = \mathfrak{M}(X, -\alpha), \quad \text{and} \quad \mathfrak{L}(X, \alpha) = \mathfrak{M}(-X, \alpha).$$

Finally for the Mellin tranform $M(f_X, \alpha)$ it holds

$$\mathfrak{L}(X, \alpha) = M(f_X(-\log x), \alpha).$$

Whereas in Aumann and Serrano (2008) the index of riskiness is calculated explicitly only for normal distributed positions, we just presented methods to calculate the indifference function, and hence dual risk measures, for positions whose integral transforms exist.

4 Representation Theorem

We present the main theorem, which states that any dual risk measure inverts the ordering, which is induced by the indifference function α. In addition, the other implication of the theorem identifies all dual risk measures and provides a criterium for duality.

Theorem 6 *A risk measure R is dual if and only if for all $X, Y \in \mathfrak{X}$ it holds*

$$R(X) > R(Y) \to \alpha(X) \leq \alpha(Y). \tag{2}$$

Proof. "If-Direction". Suppose equation (2) holds. We show that R is dual. We fix any agents i, j and gambles X, Y. W.l.o.g. we set $x_i = x_j = 0$, see footnote 12, Aumann and Serrano (2008). By assumption of equation (1), i accepts X at 0, i.e.

$$\mathbf{E}u_i(X) > 0.$$

We set $\underline{\rho} := \inf_{x\in\mathbb{R}} \rho_i(x)$ and $\overline{\rho} := \sup_{x\in\mathbb{R}} \rho_j(x)$. By $\rho_i(x) \geq \underline{\rho}$ for all x and Lemma 7 we have $u_i \leq \bar{u}_{\underline{\rho}}$, and hence

$$\mathbf{E}\bar{u}_{\underline{\rho}}(X) > 0.$$

Note that it is

$$\underline{\rho} < \alpha(X),$$

since otherwise $\underline{\rho} \geq \alpha(X)$ yields by Lemma 7 the inequality $\bar{u}_{\underline{\rho}} \leq \bar{u}_{\alpha(X)}$, which in turn yields by $\mathbf{E}\bar{u}_{\alpha(X)}(X) = 0$ the contradiction $\mathbf{E}\bar{u}_{\underline{\rho}}(X) \leq 0$.

The first assumption of equation (1) $i \rhd j$ yields with Lemma 9 the inequality $\overline{\rho} \leq \underline{\rho}$. The last assumption of equation (1) $R(X) > R(Y)$ yields by equation (2) the inequality $\alpha(X) \leq \alpha(Y)$. We sum up the upper inequalities by

$$\overline{\rho} \leq \underline{\rho} < \alpha(X) \leq \alpha(Y).$$

Together with Lemma 8 this yields $\bar{u}_{\overline{\rho}} > \bar{u}_{\alpha(Y)}$ on $\mathbb{R}^{-\{0\}}$. This yields finally by definition of $\alpha(Y)$

$$\mathbf{E}\bar{u}_{\overline{\rho}}(Y) > 0.$$

for any $Y \neq 0$. The relation $\rho_j(x) \leq \overline{\rho}$ for all x and Lemma 7 close the proof that R is dual by yielding

$$\mathbf{E}u_j(Y) > 0.$$

"Only If-Direction". The converse direction we show by the contrapositive: If equation (2) is violated, then R is not dual. Hence we have to show that there exist some agents i, j and some gambles X, Y, which satisfy the assumptions of equation (1) and moreover j does not accept Y at w_j.

W.l.o.g. we set $w_i = w_j = 0$, see footnote 12, Aumann and Serrano Footnote 12. By the violation of equation (2), there exist some gambles X, Y with $R(X) > R(Y)$ and $\alpha(X) > \alpha(Y)$. For the proof we consider these two gambles and two CARA-agents i, j with parameters

$$\rho_i = \frac{\alpha(X) + \alpha(Y)}{2},$$
$$\rho_j = \alpha(Y).$$

This choice satisfies the assumptions of equation (1). The ordering $i \rhd j$ follows by $\rho_i > \rho_j$, Lemma 9 and its contrapositive.

$$\mathbf{E}\bar{u}_{\rho_i}(X) > 0$$

follows by $\rho_i < \alpha(X)$, Lemma 8 and definition of α for any $X \neq 0$. What remains to be shown is that agent j does not accept Y at 0. By $\rho_j = \alpha(Y)$ it is

$$\mathbf{E}\bar{u}_{\rho_j}(Y) = 0,$$

which completes the proof. ■

In the proof we made use of the following three lemmata. The proofs for them are given in Aumann and Serrano (2008) and are not repeated here.

Lemma 7 *If it is $\rho_i(x) \leq \rho_j(x)$ for all x, then it holds $u_i(x) \geq u_j(x)$ for all x.*

Proof. See Aumann and Serrano (2008) Corollary 3. ■

Lemma 8 *If it is $\alpha < \beta$, then it holds $\bar{u}_\alpha(x) > \bar{u}_\beta(x)$ for all $x \neq 0$.*

Proof. See Aumann and Serrano (2008) Equation 9. ■

Lemma 9 *It is $i \unrhd j$ if and only if $\rho_i(x_i) \geq \rho_j(x_j)$ for all x_i, x_j.*

Proof. See Aumann and Serrano (2008) Proposition 3.1. ■

For formal clarity we restate the theorem in a set-theoretic notation.

Corollary 10 (Set-Theoretic Notation) *We denote* $\prec_R := \{(X,Y) \in \mathfrak{X} \times \mathfrak{X} \mid R(X) < R(Y)\}$ *the strict ordering on* $\mathfrak{X}$ *induced by* R *and* $\mathrm{id}_R := \{(X,Y) \in \mathfrak{X} \times \mathfrak{X} \mid R(X) = R(Y)\}$. *A risk measure* R *is dual if and only if*

$$\prec_R \subset \{\prec_{-\alpha} \cup \mathrm{id}_\alpha\}.$$

A direct corollary of the theorem closes the section and provides that any dual risk measure can be expressed by a composition of the indifference function and a inverting function ϕ. Moreover, the corollary provides a method to obtain dual risk measures.

Corollary 11 (Representation Theorem) *If a risk measure* R *is dual and it holds* $\mathrm{id}_\alpha \subset \mathrm{id}_R$*, then there exists a mapping* $\phi : \mathrm{im}(\alpha) \to \mathbb{R}$ *with* $\phi' \leq 0$ *and*

$$R = \phi \circ \alpha.$$

On the other hand, for any mapping $\phi : \mathbb{R} \to \mathbb{R}$ *with* $\phi' \leq 0$ *the risk measure* R*, defined by the composition*

$$R = \phi \circ \alpha$$

is dual.

5 First Examples

As shown in Corollary 11 for any mapping $\phi : \mathbb{R} \to \mathbb{R}$ with $\phi' \leq 0$ we receive by the composition of the indifference function and ϕ a risk measure $R = \phi \circ \alpha$, which respects comparative risk aversion. There are many possible choices for ϕ. Important examples are $\phi(x) = -x$, $\phi(x) = \frac{1}{x}$, $\phi(x) = -\exp(x)$ and $\phi(x) = \log(x)$. In the upcoming weeks we will consider more examples. We will check their properties of coherence and compare them to traditional risk measures. We will also show that all of them satisfy monotonicity, continuity and consistency with first and second order stochastic dominance, since these properties are induced by the indifference function α. Setting $\phi(x) = \frac{1}{x}$ yields the original index by Aumann and Serrano (2008). They derived that it satisfies monotonicity, positive homogeneity, continuity and consistency with first and second order stochastic dominance.

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